SPECIAL GROUPS AND PROJECTIVE PLANES

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Let $\mathbb{H}$ be the quaternions.

Proposition 0.1. *$SO(3)$ is diffeomorphic to $\mathbb{RP}^3$.*

Proof. Let f denote the map $q \to [v \to qvq^{-1}]$ where $q \in \mathbb{H}$, $|q| = 1$ and $v \in \Im(\mathbb{H})$, the imaginary part of $\mathbb{H}$. We want to show that f is a Lie group homomorphism from S^3 to $SO(3)$. First, note that, if $p, q \in S^3$, then

$$f(pq) = [v \mapsto (pq)v(pq)^{-1}] = [v \mapsto pqvq^{-1}p^{-1}] = [v \mapsto pvp^{-1}] \circ [v \mapsto qvq^{-1}]$$

so, assuming we can show it is well-defined, f is a group homomorphism. Now, if $q \in S^3$, then $q = (a, b, c, d)$ for some $a, b, c, d \in \mathbb{R}$, which we can re-write as $q = a + bi + cj + dk$. Now, if $v = (1, 0, 0) \in \mathbb{R}^3$, then, considered as an element of $\Im(\mathbb{H})$, $v = i$ and

$$\begin{aligned} qvq^{-1} &= (a + bi + cj + dk)i(a - bi - cj - dk) \\ &= (-b + ai + dj - ck)(a - bi - cj - dk) \\ &= i(a^2 + b^2 - c^2 - d^2) + j(2ad + 2bc) + k(2bd - 2ac) \in \Im(\mathbb{H}). \end{aligned}$$

Similarly,

$$\begin{aligned} q(j)q^{-1} &= (a + bi + cj + dk)j(a - bi - cj - dk) \\ &= (-c - di + aj + bk)(a - bi - cj - dk) \\ &= i(2bc - 2ad) + j(a^2 + c^2 - b^2 - d^2) + k(2ab + 2cd) \in \Im(\mathbb{H}) \end{aligned}$$

and

$$\begin{aligned} q(k)q^{-1} &= (a + bi + cj + dk)k(a - bi - cj - dk) \\ &= (-d + ci - bj + ak)(a - bi - cj - dk) \\ &= i(2ac + 2bd) + j(2cd - 2ab) + k(a^2 + d^2 - b^2 - c^2) \in \Im(\mathbb{H}). \end{aligned}$$

Hence, $f(q)$ is given by the matrix

$$\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2ac + 2bd \\ 2ad + 2bc & a^2 + c^2 - b^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2ab + 2cd & a^2 + d^2 - b^2 - c^2 \end{pmatrix}. \tag{1}$$

Now, we can extend this to a map from $\mathbb{R}^4$ to $M(3, 3, \mathbb{R}) = \mathbb{R}^9$, where $f(a, b, c, d)$ is simply the matrix given above; the coordinate functions of this map are clearly smooth, so f is smooth as a map $\mathbb{R}^4 \to \mathbb{R}^9$; restricting it's domain to S^3 doesn't affect its smoothness. Note that $f(1) = Id_3$ and that, if $q \in S^3$ and since $q^{-1} = a - bi - cj - dk$,

$$f(q^{-1}) = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc + 2ad & -2ac + 2bd \\ -2ad + 2bc & a^2 + c^2 - b^2 - d^2 & 2cd + 2ab \\ 2bd + 2ac & -2ab + 2cd & a^2 + d^2 - b^2 - c^2 \end{pmatrix} = f(q)^t.$$

Thus, using the result proved at the beginning of this proof,

$$Id_3 = f(1) = f(qq^{-1}) = f(q)f(q^{-1}) = f(q)f(q)^t,$$

so we see that $f(q) \in O(3)$ for all $q \in S^3$. Furthermore, since $f(1) = Id_3 \in SO(3)$, f is continuous and S^3 is connected, the image of f must be connected and, thus, must lie inside $SO(3)$. Therefore, given that we've shown that f is smooth and that f preserves the group structure, we see that

$$f : S^3 \to SO(3)$$

is a Lie group homomorphism.

Let us now calculate Ker f. If $(a, b, c, d) = q \in \text{Ker } f$, then, setting the matrix in equation (1) equal to the identity, we see that

$$\begin{aligned} a^2 + b^2 - c^2 - d^2 &= 1 \\ a^2 + c^2 - b^2 - d^2 &= 1 \\ a^2 + d^2 - b^2 - c^2 &= 1 \\ a^2 + b^2 + c^2 + d^2 &= 1, \end{aligned}$$

where the last equation comes from the fact that $q \in S^3$. The only solutions to this system of equations are $a = \pm 1$, $b = c = d = 0$, so Ker $f = \{\pm 1\}$ when viewed as a subset of the quaternions. Now, as groups,

$$S^3/(\text{Ker } f) \simeq \text{Image} f;$$

furthermore, since $\{\pm 1\}$ is a discrete subgroup of the center of S^3, we see that $S^3/(\text{Ker } f)$ is in fact a Lie group and since the above isomorphism is given by a restriction of f to this group, this is, in fact, a diffeomorphism.

Now, since Ker $f = \{\pm 1\}$, $S^3/(\text{Ker } f) = \mathbb{RP}^3$ so the image of f is diffeomorphic to $\mathbb{RP}^3$. Therefore, we need only note that f is surjective to conclude that $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$. □

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